# Solution to PurpleMind's Brainstorm Episode 2

dino

In PurpleMind's second episode of their *Brainstorm* math puzzle YouTube series, we are challenged to find a way to relabel the sides of two 8-sided dice with positive integers so that the probability distribution of their sum remains the same when they are rolled. Here I present a solution that can be used to find pairs of dice that replicate the probability distribution of the sum of common $n$-sided dice. This method basically relies on the relationship between the addition of discrete random variables and the product of polynomials; and on the use of cyclotomic polynomials.

## 1 The probability distribution of the sum of two $n$-sided dice

The probability distribution of the roll of a common $n$-sided die (labeled from 1 to $n$) is the discrete uniform distribution $\mathcal{U}(1, n)$. That is, the probability of rolling a specific number on the die is $\frac{1}{n}$.
Rolling two of these dice (and assuming that these two rolls are independent of each other), we can study the probability distribution of the sum of the rolls. In the following table we see the possible sums from the rolls of two 6-sided dice:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Since the distributions of each individual roll are uniform, so is that of a specific roll of both dice. That is, the probability of each specific roll (each cell on this table) is the same and, since there are $n^2$ cells, it is $\frac{1}{n^2}$. For example, the probability of specifically rolling a 5 on the first die and a 3 on the second die is $\frac{1}{36}$. Notice that it is important to differentiate which die is which.
Because of this, we just need to count cells to find the probability distribution of the sum of the roll and then divide by $n^2$ to turn the result into probabilities. Denoting this sum by $S$, this is the result for the 6-sided dice example:

| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(S = s)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

We could simplify these fractions. In fact, the probability of the sum of two $n$-sided dice being $n+1$ is always $\frac{1}{n}$. However, by writing it like this we can easily see the arising pattern in the distribution.

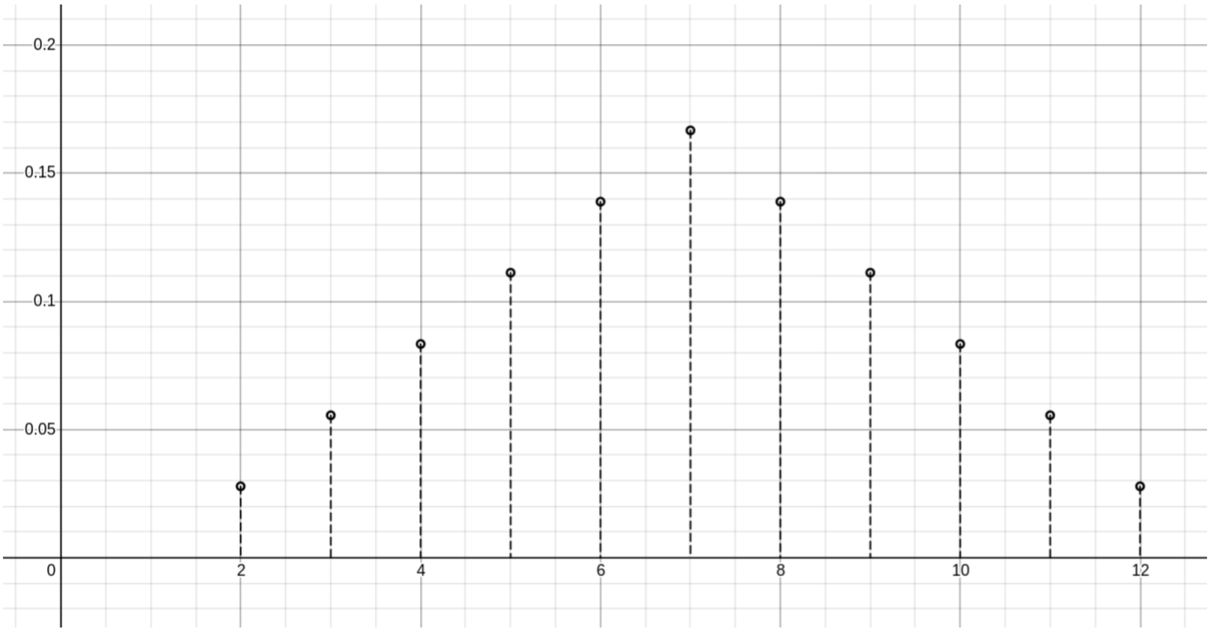


Figure 1: PMF of the sum of two 6-sided dice. Made with Desmos.

This distribution is some form of "discrete triangular distribution". The name *triangular distribution* is already used for a different continuous distribution, so this name might be confusing. Let's denote this distribution by $\mathcal{T}(n)$.
Now, in my attempt to find a solution to this problem, I first looked at a few solutions I could find. Here are the tables of some relabelings of 4-sided, 6-sided (the one given in the video) and 8-sided dice:

| | 1 | 2 | 2 | 3 |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 |
| 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 5 | 5 | 6 |
| 5 | 6 | 7 | 7 | 8 |

| | 1 | 2 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 5 |
| 3 | 4 | 5 | 5 | 6 | 6 | 7 |
| 4 | 5 | 6 | 6 | 7 | 7 | 8 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 |
| 6 | 7 | 8 | 8 | 9 | 9 | 10 |
| 8 | 9 | 10 | 10 | 11 | 11 | 12 |

| | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 |
| 3 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| 7 | 8 | 9 | 9 | 10 | 10 | 11 | 11 | 12 |
| 7 | 8 | 9 | 9 | 10 | 10 | 11 | 11 | 12 |
| 9 | 10 | 11 | 11 | 12 | 12 | 13 | 13 | 14 |
| 11 | 12 | 13 | 13 | 14 | 14 | 15 | 15 | 16 |

I found these solutions with just a bit more than brute force and then I stared at them for some hours trying to find some pattern or property that I could exploit. For example, these solutions look like the numbers follow slanted diagonals. Could we find more solutions by slanting the diagonals even more? Maybe that only works for bigger values of $n$? Do all possible solutions follow this pattern? I mainly looked at the apparent symmetries in these solutions, where we could find each cell's "complement" by rotating the table 180º:

| | 1 | 2 | 2 | 3 |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 |
| 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 5 | 5 | 6 |
| 5 | 6 | 7 | 7 | 8 |

| | 1 | 2 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 5 |
| 3 | 4 | 5 | 5 | 6 | 6 | 7 |
| 4 | 5 | 6 | 6 | 7 | 7 | 8 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 |
| 6 | 7 | 8 | 8 | 9 | 9 | 10 |
| 8 | 9 | 10 | 10 | 11 | 11 | 12 |

| | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 |
| 3 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| 5 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| 7 | 8 | 9 | 9 | 10 | 10 | 11 | 11 | 12 |
| 7 | 8 | 9 | 9 | 10 | 10 | 11 | 11 | 12 |
| 9 | 10 | 11 | 11 | 12 | 12 | 13 | 13 | 14 |
| 11 | 12 | 13 | 13 | 14 | 14 | 15 | 15 | 16 |

One of the main things I wondered about was the symmetry in the dice, where the faces (the rows and columns in the tables) are themselves "symmetrical". The dice being "symmetrical" make the distribution of their sum also "symmetrical", which is necessary for the distribution to be $\mathcal{T}(n)$. However, I wonder if there are solutions with dice faces that are not symmetrical in this sense, but such that the resulting sum distribution is still $\mathcal{T}(n)$. I couldn't prove any of this with this geometrical approach. I might have missed an elegant way to solve the problem with just this, but I think I have reached a more interesting (or unnecessarily complicated) way.

## 2 Convolutions and the moment-generating function

A more general way to find the probability distribution of a sum $S$ of two independent random variables $X$ and $Y$ is using convolutions. In the discrete case where $X$ takes values $x_1, x_2, \ldots$, we have

$$P(S=s)=\sum_{k=1}^{\infty} P(X=x_k)P(Y=s-x_k)$$

I remembered how, in the continuous case, the *convolution product* (the continuous version of this equation) behaves "nicely" along with the *bilateral Laplace transform*:

$$(f*g)(x):=\int_{-\infty}^{\infty} f(t)g(x-t)dt, \quad \mathcal{B}\{f\}(s):=\int_{-\infty}^{\infty} e^{-st}f(t)dt$$

$$\mathcal{B}\{f*g\}(s)=\mathcal{B}\{f\}(s)\,\mathcal{B}\{g\}(s)$$

I thought this could be useful, that I could easily work with the distribution of the sum of two variables as a normal product. I then realized that this is simply what the *moment-generating function* is for. In fact, the moment-generating function of a random variable is (except for a negative sign) the analogue of the bilateral Laplace transform. If $X$ is a random variable, then its moment-generating function is $M_X(t):=E\left[e^{tX}\right]$. In the discrete case, this is

$$M_X(t)=E\left[e^{tX}\right]=\sum_{k=1}^{\infty} e^{x_k t}P(X=x_k)$$

and it has that, if $X$ and $Y$ are independent,

$$M_{X+Y}(t)=M_X(t)M_Y(t)$$

In the case of the discrete uniform distribution, if $X\sim\mathcal{U}(1,n)$, then its moment-generating function is

$$M_X(t)=\sum_{k=1}^{n} e^{kt}\frac{1}{n}=\frac{1}{n}\sum_{k=1}^{n} e^{kt}$$

Then, if $Y\sim\mathcal{U}(1,n)$ too and $S=X+Y$, then

$$M_S(t)=M_X(t)M_Y(t)=\frac{1}{n^2}\left(\sum_{k=1}^{n} e^{kt}\right)^2=$$
$$=\frac{1}{n^2}\left[e^{2t}+2e^{3t}+\cdots+(n-1)e^{nt}+ne^{(n+1)t}+(n-1)e^{(n+2)t}+\cdots+2e^{(2n-1)t}+e^{2nt}\right]$$

which could be calculated directly from the definition of $M_S(t)$ and the PMF of $S\sim\mathcal{T}(n)$. Now, assume that we had a relabeling of the dice such that the first die has $x_k$ faces labeled with $k$ and the second die has $y_k$ faces labeled with $k$. Let's say that the dice have $n_X$ and $n_Y$ faces respectively. That is,

$$n_X=\sum_{k=1}^{\infty} x_k, \quad n_Y=\sum_{k=1}^{\infty} y_k$$

Then, the moment-generating function of rolls $X$ and $Y$ would be

$$M_X(t) = \frac{1}{n_X} \sum_{k=1}^{\infty} x_k e^{kt}, \quad M_Y(t) = \frac{1}{n_Y} \sum_{k=1}^{\infty} y_k e^{kt}$$

In order for their sum $S = X+Y$ to follow the $\mathcal{T}(n)$ distribution, the product of these two functions must be the moment-generating function from before. Therefore, the problem of finding these dice is the same as factoring that moment-generating function into two other valid moment-generating functions.

## 3 Using polynomials

Turns out that these moment-generating functions behave pretty much like polynomials do. If we substitute $e^t$ with $x$ in the moment-generating function of $\mathcal{T}(n)$ and ignore the $\frac{1}{n^2}$ coefficient, we obtain

$$\left(\sum_{k=1}^{n} x^k\right)^2 = x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1} + (n-1)x^{n+2} + \cdots + 2x^{2n-1} + x^{2n}$$

We can also divide by $x^2$ to simplify things. This turns out to be the same as considering the dice to allow faces labeled with 0 (non-positive integers) and considering that a common die has faces going from 0 to $n-1$. We obtain the polynomial

$$T_n^2(x) := \left(\sum_{k=0}^{n-1} x^k\right)^2 = 1+2x+\cdots+(n-1)x^{n-2}+nx^{n-1}+(n-1)x^n+\cdots+2x^{2n-3}+x^{2n-2}$$

We are associating this polynomial to a discrete random variable $X$ where each term $x^k$ corresponds to the occurrence of $X = k+1$ with probability proportional to that term's coefficient. For example, we associate the polynomial $T_n(x) = \sum_{k=0}^{n-1} x^k$ with the distribution $\mathcal{U}(1, n)$ and a common $n$-sided die.
This means that in order to find a new couple of dice that simulates $\mathcal{T}(n)$, we can factorize $T_n^2(x)$ into two polynomials and find their associated distributions. That is, as long as the two polynomials have non-negative integer coefficients. Otherwise, they wouldn't correspond to actual probability distributions or to any die.
On top of this, the resulting die from factorizing $T_n^2(x)$ aren't necessarily $n$-sided. The number of faces in each die corresponds to the sum of the coefficients in each factor. For example, in the $n = 4$ case, we could consider the trivial factorization:

$$T_4^2(x) = (1 + x + x^2 + x^3)^2 = (1 + x + x^2 + x^3)(1 + x + x^2 + x^3) = P_X(x)P_Y(x)$$

We have factorized $T_4^2(x)$ into the polynomials $P_X(x) = P_Y(x) = 1 + x + x^2 + x^3$. This means that both dice $X$ and $Y$ have each

- 1 face labeled with a 1. (corresponding to the term 1)
- 1 face labeled with a 2. (corresponding to the term $x$)
- 1 face labeled with a 3. (corresponding to the term $x^2$)
- 1 face labeled with a 4. (corresponding to the term $x^3$)

This yields the usual pair of 4-sided dice.
Let's see another way to factorize $T_4^2(x)$:

$$T_4^2(x) = (1 + x + x^2 + x^3)^2 = (1 + 2x + x^2)(1 + 2x^2 + x^4) = P_X(x)P_Y(x)$$

The polynomial $P_X(x) = 1 + 2x + x^2$ corresponds to the die $X$ with

- 1 face labeled with a 1. (corresponding to the term 1)
- 2 faces labeled with a 2. (corresponding to the term $2x$)

- 1 face labeled with a 3. (corresponding to the term $x^2$)

while the polynomial $P_Y(x) = 1 + 2x^2 + x^4$ corresponds to the die $Y$ with

- 1 face labeled with a 1. (corresponding to the term 1)
- 2 faces labeled with a 3. (corresponding to the term $2x^2$)
- 1 face labeled with a 5. (corresponding to the term $x^4$)

These are the dice in the previous table for the $n = 4$ case. For the other two tables with non-trivial solutions, we had the following factorizations:

$$T_6^2(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^2 = (1 + 2x + 2x^2 + x^3)(1 + x^2 + x^3 + x^4 + x^5 + x^7)$$

$$T_8^2(x) = (1+x+x^2+x^3+x^4+x^5+x^6+x^7)^2 = (1+2x+2x^2+2x^3+x^4)(1+x^2+2x^4+2x^6+x^8+x^{10})$$

As we said, the number of faces of each die is the sum of the coefficients in its polynomial, so it may not be $n$-sided. A clear case of this is the also trivial factorization given by

$$T_n^2(x) = 1 \cdot T_n^2(X)$$

where $P_X(x) = 1$ means that the $X$ die is a 1-sided die labelled with a 1, with only one outcome; while the $Y$ die is a $n^2$-sided die. For the $n = 4$ case, we have

$$T_4^2(x) = 1 \cdot (1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6)$$

which returns the table

| | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 7 | 7 | 8 |

A less trivial example could be

$$T_4^2(x) = (1 + x)(1 + x + 2x^2 + 2x^3 + x^4 + x^5)$$

which gives

| | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 5 | 6 | 6 | 7 | 8 |

This all feels like an over-complicated mess, but it does give some interesting insight. Attempting to find relabelings of the dice by hand, I couldn't find any for the cases $n = 3, 5$ or 7. I suspected that maybe odd values of $n$ didn't have non-trivial relabelings, or that at least we would have to look at bigger values of $n$ to find some. Maybe this is why PurpleMind asks us to find relabelings for $n = 8$ after showing the $n = 6$ case and skipping $n = 7$. The actual reason may not be because 7-sided dice are not easy to come by, but is actually somewhat interesting.
The polynomial $T_n(x) = \sum_{k=1}^{n} x^k$ turns out to be irreducible for prime values of $n$. You can find a proof of this using Eisenstein's criterion in the Wikipedia. This means that, much like the square of a prime number, the polynomial $T_n^2(x)$ cannot be factorized in

any other way than $T_n(x) \cdot T_n(x)$ and $1 \cdot T_n^2(x)$. Otherwise we would find a divisor of $T_n(x)$ and it wouldn't be irreducible. This means that there are no non-trivial relabelings for the cases where $n$ is prime.
Furthermore, this method could be used to generate any amount of dice such that, when they are all rolled, their sum simulates the $\mathcal{T}(n)$ distribution. To do this, we could factorize $T_n^2(x)$ into three or more factors. Similarly, it is possible that factorizing $T_n^m(x)$ for higher values of $m$ generalizes this problem to the replication of the roll of $m$ $n$-sided dice.

# 4 More on cyclotomic polynomials

Up until now I believed that a *cyclotomic polynomial* was a polynomial of the form of $T_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$. I hadn't realized this mistake before since I haven't used cyclotomic polynomials properly until now. Even though the cyclotomic polynomials are not exactly $T_n(x)$ as we will see in a moment, these polynomials are still very useful to us. We have that

$$x^n - 1 = (x-1)(1 + x + x^2 + \cdots + x^{n-1})$$

The polynomial $x^n - 1$ has the *n-th roots of unity* as its $n$ roots. These roots of unity are the complex numbers whose $n$-th power is 1. For example, $-1$ is a second root of unity since $(-1)^2 = 1$ and in fact, it is an $n$-th root of unity for any even $n$. Aside from 1 and $-1$, $i$ and $-i$ are the other two fourth roots of unity, since $i^4 = (-1)^4 = 1$. An easy way to visualize the $n$-th root of unities is to draw the complex unit circle and inscribe an $n$-sided polygon in it such that it has a vertex at 1. Then, the $n$-th roots of unity are the vertices of this polygon. Since $(x^n - 1) = (x-1)T_n(x)$, we know that the $n-1$ roots of $T_n(x)$ are the $n$-th roots of unity excluding 1.
The $n$-th cyclotomic polynomial $\phi_n(x)$ is similar to $T_n(x)$ in that its roots are also $n$-th roots of unity, but they are not necessarily all the $n$-th roots. Its roots are the *primitive* $n$-th roots of unity, which are the ones such that they are not roots of another polynomial $x^m - 1$ with $m < n$. We can write the $n$-th roots of unity as

$$e^{\frac{2\pi ik}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 0, \ldots, n-1$$

Then, we could show that the root corresponding to $k$ is primitive if and only if $k$ and $n$ are coprime. This means that, if $n$ is prime, $\phi_n(x)$ and $T_n(x)$ do coincide. Another important property (or equivalent definition) of $\phi_n(x)$ is that it is the unique irreducible polynomial with integer coefficients that divides $x^n - 1$. In fact, we have that

$$x^n - 1 = \prod_{d|n} \phi_d(x)$$

This is very important, since it gives some kind of unique "prime factorization" for $x^n - 1$ and therefore for $T_n(x)$:

$$T_n(x) = \prod_{\substack{d|n \\ d\neq 1}} \phi_d(x)$$

Now, this lets us more easily control the number of faces in each die. Since this number is the sum of the coefficients in the factor polynomials, it is $P_X(1)$ and $P_Y(1)$. That is, the polynomials evaluated at $x = 1$. We can easily see that

$$T_n^2(1) = P_X(1)P_Y(1) = \prod_{\substack{d|n \\ d\neq 1}} \phi_d(1)$$

These cyclotomic polynomials have one last interesting property that lets us control this number of faces. They have that

$$\phi_n(1) = \begin{cases} p & \text{if } n \text{ is a power of some } p \text{ prime} \\ 1 & \text{if not} \end{cases}$$

This lets us easily find factorizations of $T_n^2(x)$ such that the resulting dice are $n$-sided. For example, for $n = 4$, we have

$$T_4^2(x) = \phi_2^2(x)\phi_4^2(x), \quad T_4^2(1) = 2^2 \cdot 2^2 = 16$$

Because 4 is $2^2$, all the cyclotomic polynomials involved evaluate to 2 at $x = 1$ so in order to find a factorization for 4-sided dice, we have to pick two of these polynomials for $P_X(x)$. The options are $P_X(x) = \phi_2(x)\phi_4(x) = T_4(x)$, which is the trivial one; or $P_X(x) = \phi_2^2(x) = 1 + 2x + x^2$ which is the one found before. Notice that $P_X(x) = \phi_4^2(x)$ is just the same case as $P_X(x) = \phi_2^2(x)$, since it would just correspond to the other die and the cases are symmetrical.
Let's see now how this works for $n = 6$. This is the first case where we have a composite divisor that is not a power of a prime. We have that

$$T_6^2(x) = \phi_2^2(x)\phi_3^2(x)\phi_6^2(x), \quad T_6^2(1) = 2^2 \cdot 3^2 \cdot 1^2 = 36$$

We notice that, in order to have $P_X(1) = 6$, the factorization of $P_X(x)$ must include $\phi_2(x)\phi_3(x)$ and no more copies of $\phi_2(x)$ and $\phi_3(x)$. The only leeway we get in this case is whether $P_X(x)$ includes (has a divisor) $\phi_6(x)$ or not. (The case where $\phi_6^2(x)$ is included is symmetrical to the case where $P_X(x) = \phi_2(x)\phi_3(x)$.)
Therefore, we have only two cases: The trivial $P_X(x) = \phi_2(x)\phi_3(x)\phi_6(x) = T_6(x)$ and the non-trivial $P_X(x) = \phi_2(x)\phi_3(x) = 1 + 2x + 2x^2 + 1$ found previously.
And one more time for $n = 8$: We have

$$T_8^2(x) = \phi_2^2(x)\phi_4^2(x)\phi_8^2(x), \quad T_8^2(1) = 2^2 \cdot 2^2 \cdot 2^2 = 64$$

This time, we have much more freedom to choose the polynomials, since the only requisite seems to be that $P_X(x)$ is the product three of these cyclotomic polynomials. I believe the only four possible cases we have are:

- $P_X(x) = \phi_2(x)\phi_4(x)\phi_8(x)$, the trivial case.
- $P_X(x) = \phi_2^2(x)\phi_4(x)$, the non-trivial case found previously.
- $P_X(x) = \phi_2^2(x)\phi_8(x)$, which has

$$P_X(x) = 1 + 2x + x^2 + x^3 + 2x^4 + x^6$$

$$P_Y(x) = 1 + 2x^2 + 2x^4 + 2x^6 + x^8$$

| | 1 | 2 | 2 | 3 | 5 | 6 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 6 | 7 | 7 | 8 |
| 3 | 4 | 5 | 5 | 6 | 8 | 9 | 9 | 10 |
| 3 | 4 | 5 | 5 | 6 | 8 | 9 | 9 | 10 |
| 5 | 6 | 7 | 7 | 8 | 10 | 11 | 11 | 12 |
| 5 | 6 | 7 | 7 | 8 | 10 | 11 | 11 | 12 |
| 7 | 8 | 9 | 9 | 10 | 12 | 13 | 13 | 14 |
| 7 | 8 | 9 | 9 | 10 | 12 | 13 | 13 | 14 |
| 9 | 10 | 11 | 11 | 12 | 14 | 15 | 15 | 16 |

- $P_X(x) = \phi_2(x)\phi_4^2(x)$, which has

$$P_X(x) = 1 + x + 2x^2 + 2x^3 + x^4 + x^5$$

$$P_Y(x) = 1 + x + 2x^4 + 2x^5 + x^8 + x^9$$

| | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 5 | 6 | 6 | 7 | 8 |
| 5 | 6 | 7 | 8 | 8 | 9 | 9 | 10 | 11 |
| 5 | 6 | 7 | 8 | 8 | 9 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 9 | 10 | 10 | 11 | 12 |
| 6 | 7 | 8 | 9 | 9 | 10 | 10 | 11 | 12 |
| 9 | 10 | 11 | 12 | 12 | 13 | 13 | 14 | 15 |
| 10 | 11 | 12 | 13 | 13 | 14 | 14 | 15 | 16 |

Now, by using the cyclotomic polynomials, we ensure that the coefficients of $P_X(x)$ and $P_Y(x)$ are integers. However, some cyclotomic polynomials have negative integer coefficients, which makes some products of cyclotomic polynomials also have negative coefficients and, therefore, not valid solutions. We haven't found any case like this so far and I suspect that, somehow, imposing $P_X(1) = n$ forces all coefficients to be non-negative.
Assuming that this is true, we reach another interesting piece of insight about this problem. Once we fix $n$, the "structure" of the solutions only depends on the "factorization structure" of $n$. That is, for example, whether $n$ is the product of three primes or the square of a prime or something else. The fact that all problems with $n$ prime have no non-trivial solutions is a consequence of this. This structure is known as the *prime signature* of $n$.
We have found solutions for $n = 4, 6$ and 8. 4 is a square of a prime; 6 is the product of two primes; and 8 is the cube of a prime. Therefore, we already know which solutions to look for whenever we look at higher numbers $n$ with this signature. For example, for $n = 9 = 3^2$, we will have only two answers: the trivial one and the non-trivial one, which will be

$$P_X(x) = \phi_3^2(x) = 1 + 2x + 3x^2 + 2x^3 + x^4$$
$$P_Y(x) = \phi_9^2(x) = 1 + 2x^3 + 2x^6 + 2x^9 + x^{12}$$

| | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 6 |
| 4 | 5 | 6 | 6 | 7 | 7 | 7 | 8 | 8 | 9 |
| 4 | 5 | 6 | 6 | 7 | 7 | 7 | 8 | 8 | 9 |
| 7 | 8 | 9 | 9 | 10 | 10 | 10 | 11 | 11 | 12 |
| 7 | 8 | 9 | 9 | 10 | 10 | 10 | 11 | 11 | 12 |
| 7 | 8 | 9 | 9 | 10 | 10 | 10 | 11 | 11 | 12 |
| 10 | 11 | 12 | 12 | 13 | 13 | 13 | 14 | 14 | 15 |
| 10 | 11 | 12 | 12 | 13 | 13 | 13 | 14 | 14 | 15 |
| 13 | 14 | 15 | 15 | 16 | 16 | 16 | 17 | 17 | 18 |

# 5 Conclusion

We have seen a method of generating pairs of dice with relabeled faces that replicate the roll of two common $n$-sided dice. We have done so through the study of polynomials, specifically of the polynomials $T_n^2(x)$. Through the use of cyclotomic polynomials, we have shown that this problem is reduced to choosing the appropriate polynomials out of the factorization of $T_n^2(x)$.
This has shown that the form of the solutions and the number of the solutions depends only on the prime signature of $n$. By finding the solutions of $n = 4, 6$ and 8, we know how to find the solutions for any $n$ that is a square or cube of a prime or a product of two primes. Though this is all under the assumption that the resulting polynomials $P_X(x)$ and $P_Y(x)$ have positive coefficients, which we have not shown.
It is interesting to note that all the example solutions we have found are symmetrical, just like we noted at the start. It would also be interesting to prove if this is true for all possible solutions and for all values of $n$.

# 6 Addendum

Looking more into this problem, we find that it has already been studied properly by people who actually know what they're doing, though they all seem to work with cyclotomic in the same way I have. The non-trivial solution for the $n = 6$ case, the dice that Purplemind shows in their video, is known as the *Sicherman dice*. Here are a few links about this problem:

- Wikipedia article for Sicherman dice: https://en.wikipedia.org/wiki/Sicherman_dice
- OEIS sequence A111588 with the number of solutions for the $n$ case: https://oeis.org/A111588
- Free-access article by Gallian and Rusin that's basically this but better: https://doi.org/10.1016/0012-365X(79)90161-4