A Bit About Pascal's Triangle

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1 Introduction

I like Pascal's triangle. A lot. I learned about it when I was very young and slowly learned more and more little bits of trivia and properties that surround it. It is a very simple construction that 10-year-olds can easily understand while, at the same time, hiding some very interesting mathematical secrets. It's also a great pastime when your math classes get too boring.

Because of this, I've spent a decent amount of time thinking about it and trying to come up with more properties by myself. I've even tried experimenting with the triangle itself, altering it to see what happens.

Of course, you can find all of this and more somewhere else on the internet, at the Wikipedia article about Pascal's triangle, or at an actual math paper. Also, it's worth noting that some of these things are just trivia: They might be curious, but not very mathematically interesting. I'm also not aiming for perfect mathematical rigor, since I'm just a crank with access to the internet.

2 Building Pascal's triangle

I'm always a bit surprised when someone tells me they don't know about Pascal's triangle, since it is very simple to build. It consists of two simple steps:

1. Start by writing down a one.
2. To write a new number, add the two numbers above it.

Assume that everything "outside the triangle" is covered in zeroes and keep writing numbers until you reach the end of the naturals (or to where you consider appropriate). If you didn't do anything wrong, you should get something like this:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\vdots
\end{array}
$$

Figure 1: The first few rows of Pascal's triangle

You can easily see how the 2 is a 2 because it has two ones above it, or how you got all ones in the sides of the triangle, as they only have a single one above them.

That's how you build Pascal's triangle recursively. If you want to know what the 100th row of the triangle looks like, you have to start from the top and do all the math. That's a lot of math, and mathematicians hate doing math.

Luckily, some mathematicians have been somewhat smart about this and found a way to calculate a specific number in Pascal's triangle without having to do all those calculations. If you know a bit of combinatorics or have studied *Newton's binomial* or the *binomial expansion*, you may already know the answer: The number $P_{n,k}$ in the nth row and the kth position in that row is given by the formula

$$P_{n,k} := \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1}$$

The expression $\binom{n}{k}$ is the number of k-combinations of a set of n elements. If that sounds like gibberish, a k-combination is a possible selection of k things out of n things, regardless of the order they've been selected. With this in mind, $\binom{4}{2} = \frac{4!}{2!2!} = 6$ because there are 6 different ways to choose 2 things out of 4.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\vdots
\end{array}$$

Figure 2: Pascal's triangle with binomial coefficients

It's important to note that the triangle starts at the 0th row and each row starts with the 0th position. Remember that, for a lot of very good reasons, $0! = 1$. If you don't remember or know what $n!$ means, it is the *factorial* of n, which is the product of all integers bellow n and n itself:

$$n! := n \times (n-1) \times (n-2) \cdots 3 \times 2 \times 1, \quad n \in \mathbb{N}$$

With this definition, we can only take the factorials of natural numbers, but it is possible to extend it to (almost) every real number with the gamma function: $n! = \Gamma(n+1)$

The convention that $0! = 1$ seems to contradict this definition a bit, since you may think that the product in the definition would be 0. However, the sequence of factors in that product is not the sequence made of a single 0, but an empty sequence, since we go down until the 1. The product of all the factors in an empty sequence is the *empty product*[1],

[1]This is similar to the *empty intersection* or *nullary intersection*: The intersection of no sets. By definition, this would be composed of the elements x such that $x \in A \; \forall A \in \emptyset$, which is a vacuous truth, so every x is in this intersection. This would only work if the *universe set* existed, which it doesn't. But it does make sense when talking in the context of a total set X, so $\bigcap = X$

which is the multiplicative identity by convention, which is 1.

The recurrence relation with which we built the triangle (Step 2 in the instructions above) can then be written as:

$$P_{n,k} = P_{n-1,k} + P_{n-1,k-1}, \quad n \geq 1, \ 1 \leq k < n \tag{2}$$

We can see that this is true through eq. (1) and some fraction operations:

$$\begin{aligned} P_{n-1,k} + P_{n-1,k-1} \binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} = \\ &= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!} = \frac{(n-1)!n}{k!(n-k)!} = \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k} = P_{n,k} \quad \square \end{aligned}$$

You can already notice the first property of the triangle: Symmetry! Pascal's triangle is perfectly symmetrical along its vertical axis. We can see this is true in eq. (1):

$$P_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} = P_{n,n-k} \tag{3}$$

3 Some properties, applications and trivia

3.1 Newton's binomial

Pascal's triangle has a surprising number of properties as dozens of different math concepts pop up left and right in it if you look hard enough. Here I have compiled a few of them.

I have already mentioned Newton's binomial and the binomial expansion. This is the expansion of the power of a sum $(a+b)^n$ for natural values of n. Famously and sadly, it doesn't always[2] evaluate to $a^n + b^n$ since exponentiation doesn't distribute over addition, but over multiplication. Instead, if you calculate all the products you reach the following expression:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n} a^0 b^n \tag{4}$$

This result is also known as the *binomial theorem*. For the first values of n, we get:

$$\begin{aligned} (a+b)^0 &= 1a^0b^0 \\ (a+b)^1 &= 1a^1b^0 + 1a^0b^1 \\ (a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2 \end{aligned}$$

[2]Note the "always". Sometimes it does happen that $(a+b)^n = a^n + b^n$. Of course it happens when either a or b are zero. It also happens when we work with something called *fields of prime characteristic*, fields where a prime number $p > 0$ happens to satisfy that $(a+b)^p = a^p + b^p$. The naturals and the integers and these classic sets of numbers are fields, but they have characteristic 0, so this doesn't apply. Mathematicians, with their particular humor, call this *Freshmans' dream*. Look it up.

$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$
$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$\vdots$$

Can you see Pascal's triangle? This way, we can use Pascal's triangle to easily compute a binomial expansion without ever memorizing any formula. Get to the nth row on the triangle and multiply every number by a and b, with their powers falling and raising as you go through the row.

One important property in combinatorics is that the sum of all k-combinations out of n elements is 2^n:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{5}$$

This number is the number of all possible ways to choose any amount of things out of n things, including not choosing anything at all ($k = 0$). It's also the number of possible subsets of a set A with $|A| = n$ (That's the same thing but in math jargon. For more complicated math jargon, we say $|\mathcal{P}(A)| = 2^n$ where $\mathcal{P}$ denotes the *power set*, but we'll use this notation for something else later, so forget about it).

We can easily prove that this is true with our newly-acquired binomial expansion powers. We just substitute with $a = b = 1$ in equation (4) and we're done:

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \quad \square$$

We see this reflected in Pascal's triangle when we add up each row. You'll notice we get the powers of 2:

$$1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 + 1 = 4$$
$$1 + 3 + 3 + 1 = 8$$
$$1 + 4 + 6 + 4 + 1 = 16$$
$$1 + 5 + 10 + 10 + 5 + 1 = 32$$
$$\vdots$$

You can try and find some cooler math trivia by yourself. Try adding up the diagonals as if you were a knight in chess! Or coloring only the odd numbers!

3.2 Sines and cosines

This section is a bit more complicated. It exploits Newton's binomial to generalize some known trigonometric identities. Integrating $\cos x$ is a simple task, but integrating $\cos^4 x$ or $\sin^5 x$ might involve some tedious substitutions and integration by parts. However, it is possible to write $\cos^n x$ or $\sin^n x$ as sums of easily integrable summands of the form $\cos(mx)$ for any $n \in \mathbb{N}$. In fact, it is pretty easy with the help of Pascal's triangle.

To see this, we'll first write $\cos x$ with its "complex exponential" definition:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

If you're not familiar with this expression but know a bit of complex numbers, you might be able to guess what this is doing with the help of Euler's (one of many) identity:

$$e^{ix} = \cos x + i \sin x$$

Now, let's expand $\cos^n x$ for $n \in \mathbb{N}$:

$$\begin{aligned}\cos^n x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^n = \frac{1}{2^n}(e^{ix} + e^{-ix})^n = \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{kix} e^{-(n-k)ix} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{(2k-n)ix} = \\ &= \frac{1}{2^n}\left[\sum_{k=0}^{n}\binom{n}{k}\cos\left[(2k-n)x\right] + i\sum_{k=0}^{n}\binom{n}{k}\sin\left[(2k-n)x\right]\right] \stackrel{*}{=}\end{aligned}$$

Now, notice that in the sum of sines, every sine $\sin\left[(2k-n)x\right]$ has a corresponding sine with the opposite argument $\sin\left[-(2k-n)x\right]$, so this sum is always 0 (When n is even, there's one sine that doesn't have a partner, but it's $\sin(0x) = 0$, so the result is the same). We then have that

$$\cos^n x \stackrel{*}{=} \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left[(n-2k)x\right] \tag{6}$$

In a similar fashion, we can obtain the sine equivalent, which is a bit more complicated:

$$\sin^n x = \begin{cases} \dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}(-1)^{\frac{n}{2}-k}\cos\left[(n-2k)x\right] & \text{if } n \text{ is even} \\ \dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}(-1)^{\frac{n-1}{2}-k}\sin\left[(n-2k)x\right] & \text{if } n \text{ is odd} \end{cases} \tag{7}$$

Now, because of the symmetry of Pascal's triangle and the parity of sines and cosines, this sum can be simplified a bit. To make the notation easier, we'll define Pascal's *half triangle*, which is just one side of Pascal's triangle split vertically right through its middle. The numbers that are exactly in the middle are halved too:

$$H_{n,k} = \begin{cases} \frac{1}{2}P_{n,k} & \text{if } n \text{ is even and } k = \frac{n}{2} \\ P_{n,k} & \text{if not} \end{cases} \tag{8}$$

$$
\begin{array}{rrr|rrr}
 & & \frac{1}{2} & \frac{1}{2} & & \\
 & & 1 & & 1 & \\
 & 1 & 1 & 1 & 1 & \\
 1 & 3 & & & 3 & 1 \\
 1 & 4 & 3 & 3 & 4 & 1 \\
 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\qquad
\begin{array}{rrr|}
 & & \frac{1}{2} \\
 & & 1 \\
 & 1 & 1 \\
 & 1 & 3 \\
 1 & 4 & 3 \\
 1 & 5 & 10 \\
\end{array}
\qquad
\begin{array}{rrr|}
 & & H_{0,0} \\
 & & H_{1,0} \\
 & H_{2,0} & H_{2,1} \\
 & H_{3,0} & H_{3,1} \\
 H_{4,0} & H_{4,1} & H_{4,2} \\
 H_{5,0} & H_{5,1} & H_{5,2} \\
\end{array}
$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Figure 3: Splitting Pascal's triangle in half

Let $m = \frac{n}{2}$ if n is even and $m = \frac{n-1}{2}$ if n is odd. Then, equations (6) and (7) turn into

$$\cos^n x = \frac{1}{2^{n-1}} \sum_{k=0}^{m} H_{n,k} \cos\left[(n-2k)x\right] \tag{9}$$

$$\sin^n x = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{k=0}^{m} H_{n,k} (-1)^{m-k} \cos\left[(n-2k)x\right] \\ \dfrac{1}{2^{n-1}} \displaystyle\sum_{k=0}^{m} H_{n,k} (-1)^{m-k} \sin\left[(n-2k)x\right] \end{cases} \tag{10}$$

For example, we have that

$$\cos^2 x = \frac{1}{2}\left[\cos(2x) + \cos(0x)\right] = \frac{1}{2}\cos(2x) + \frac{1}{2}$$

$$\sin^5 x = \frac{1}{2^4}\left[\sin(5x) - 5\sin(3x) + 10\sin x\right] = \frac{1}{16}\sin(5x) - \frac{5}{16}\sin(3x) + \frac{5}{8}\sin x$$

Integrating these expressions can be much easier. Now, we can find a reverse version of this trick: Turning $\cos(nx)$ into summands like $\cos^m x$ or $\cos^a x \sin^b x$. To do this, this time we expand the binomial in Euler's identity:

$$\cos(nx) + i\sin(nx) = e^{nix} = \left(e^{ix}\right)^n = (\cos x + i\sin x)^n = \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k} x \sin^k x \tag{11}$$

To continue, notice how the terms of this sum go around the unit circle, being multiplied by $1, i, -1, -i, 1, i, -1, -i, \ldots$ We can then extract $\cos(nx)$ and $\sin(nx)$ of this sum by taking the real and imaginary parts of it respectively. That would make the terms alternate between the sine and cosine, where the cosine corresponds to the terms with k even and the sines have k odd. Writing this with the sum notation can be a bit messy, so let's see it in a table and some examples:

$$
\begin{aligned}
\cos(2x) &= \cos^2 x - \sin^2 x \\
\sin(2x) &= 2\cos x \sin x \\
\cos(5x) &= \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x \\
\sin(4x) &= 4\cos^3 x \sin x - 4\cos x \sin^3 x
\end{aligned}
$$

	+	+	−	−	+	+
	c	s	c	s	c	s
$n = 0$	1					
$n = 1$	1	1				
$n = 2$	1	2	1			
$n = 3$	1	3	3	1		
$n = 4$	1	4	6	4	1	
$n = 5$	1	5	10	10	5	1

$\vdots$

Table 1: Coefficients of eq. (11). The columns alternate between corresponding to cosines (c) and sines (s). Every two columns, the sign of the coefficients also alternates.

Notice how we see Pascal's triangle again, but this time all the numbers are moved to the left. This way of writing the triangle can be easier to handle.

3.3 More triangles: Triangular numbers and n-simplex numbers

Let's look at something more geometrical now. Let's focus on the diagonals of the triangle (or columns, depending on how you see it). Each diagonal gives a sequence of numbers: The first diagonal is full of ones; the second is the natural numbers; and the third one you might recognize.

For convenience let's call the first diagonal the zeroth one instead. Using eq. (1), we can see that the mth number in the nth diagonal (starting from $m = 1$) is given by the following expression:

$$S_{n,m} = \binom{n+m-1}{n} = \frac{(n+m-1)!}{n!(m-1)!} \tag{12}$$

For $n = 0$, we get $S_{0,m} = \frac{(m-1)!}{(m-1)!} = 1$; and for $n = 1$ we have $S_{1,m} = \frac{m!}{(m-1)!} = m$, as we expected. For $n = 2$, we have

$$S_{2,m} = \frac{(m+1)!}{2!(m-1)!} = \frac{m(m+1)}{2} = \frac{m^2+m}{2} \tag{13}$$

And this formula you might recognize. It is the formula for the *triangular numbers* (the amount of dots in a triangle with a base of m dots), or the sum of the first m natural numbers. Legend says that Gauss himself discover this expression by himself as a kid, when his schoolteacher got their pupils to add all numbers from 1 to 100 and little Gauss had the solution just a minute later.

Of course, the formula probably was already well-known before that, but it's not unbelievable that a kid (specially one like Gauss) could have come up with it. The story says that Gauss imagine each number as a row of dots, forming a triangle with a hundred dots on its sides and base. He then imagined a second triangle which, upside down, fused with the first one to create a rectangle with 100 dots on one side and 101 on the other. This rectangle then had $100 \times 101 = 10100$ dots, so the original triangle must have had 5050 dots, and that was Gauss' answer. You can easily check that this is the right number by plugging $m = 100$ in eq. (13) (And you could find this number in Pascal's triangle's second diagonal if you look far enough).

$$
\begin{array}{c}
S_{0,1} \\
S_{0,2} \quad S_{1,1} \\
S_{0,3} \quad S_{1,2} \quad S_{2,1} \\
S_{0,4} \quad S_{1,3} \quad S_{2,2} \quad S_{3,1} \\
S_{0,5} \quad S_{1,4} \quad S_{2,3} \quad S_{3,2} \quad S_{4,1} \\
S_{0,6} \quad S_{1,5} \quad S_{2,4} \quad S_{3,3} \quad S_{4,2} \quad S_{5,1} \\
\vdots
\end{array}
$$

Figure 4: Pascal's triangle with $S_{n,m}$

We'll write $S_n := \{S_{n,m}\}_m$ as the sequence that we find in the nth diagonal. S_0 is a constant sequence of 1s; S_1 are the natural numbers; and S_2 are the triangular numbers. Let's study S_3. We have that

$$S_{3,m} = \frac{(m+2)!}{3!(m-1)!} = \frac{m(m+1)(m+2)}{6} = \frac{m^3 + 3m^2 + 2m}{6} \tag{14}$$

The first terms of this sequence are $1, 4, 10, 20, 35, \ldots$ Maybe you already know about this sequence or you've figured it out from the way Pascal's triangle is built, but these are the *tetrahedral numbers*, and are the amount of dots in a tetrahedron (a triangular pyramid) with an edge of n dots.

Have you noticed that S_2 corresponds to triangles, a 2D shape; while S_3 corresponds to tetrahedrons, a 3D shape? Tetrahedrons are also pretty much triangles but "in 3 dimensions", right? What about S_1 and S_0? Are they triangles "in 1 and 0 dimensions"?

In 1 dimension, we can only have lines. A line of length n dots has a total of n dots, obviously. This is what S_1 says. In 0 dimensions, it's not so obvious, but we can think of a shape in 0 dimensions (and the whole space) as a singular point. Anything more would need more dimensions, right? Then, a shape in 0 dimensions, regardless of "the length of its edge" (though it doesn't even have an edge) has only 1 dot, which is what S_0 says. If you're not happy with this 0-dimensional thinking, just think this is convention and let it be.

So, are the rest of S_n related to higher-dimensional triangles? Yes, they are. S_4 corresponds to the 4D triangles: the *pentachora*, plural for *pentachoron* . Shapes in 4D space are called *polychora*, where *-choron* means “space” in Greek, so a pentachoron has five faces, which are all tetrahedra, the same way a tetrahedron has four triangular faces. For a bit of fun, draw a pentachoron and find the five pyramids. You can easily draw a pentachoron simply by drawing all the lines joining five points, though it’s not a perfect drawing and you’ll have to squeeze your brain a bit to see the perspective.

S_5 corresponds to the 5D triangles. A name for this shape is the *hexateron*. A general name for 5D shapes is *polytera*, where *-teron* simply comes from *tetra*: four, the dimension of its faces.

The general name for these shapes is *simplex* or *n-simplex*. An n-simplex is the n-dimensional analog of the triangle. The name arrives from the fact that they’re the simplest *n-polytope* (general name for the n-dimensional analogs of polygons and polyhedra) one can build in n dimensions. They’re the convex envelope of $n+1$ (linearly independent) points (This means that they’re the shape that arises when you join $n+1$ points drawing all the lines between them). We are using the notation $S_{n,m}$ because it denotes the amount of dots in an n-simplex with an edge of length m dots. We can call the sequence S_n the sequence of *n-simplex numbers*.

To go from $S_{n,m}$ to $S_{n,m+1}$, we’re augmenting the n-simplex by adding to its base one more face with an edge of $m+1$ dots. To see this, picture it for $n = 1, 2$ and 3:

- To go from $S_{1,m}$ to $S_{1,m+1}$, we simply add one more dot ($S_{0,m+1}$) to it.
- To go from $S_{2,m}$ to $S_{2,m+1}$, we add a line of $m+1$ dots ($S_{1,m+1}$) to the base of the triangle.
- To go from $S_{3,m}$ to $S_{3,m+1}$, we add a triangle of side $m+1$ dots ($S_{2,m+1}$) to the base of the pyramid.

This can be written as:

$$S_{n,m+1} = S_{n,m} + S_{n-1,m+1}, \quad n \geq 1 \tag{15}$$

It’s very simple to prove this just by translating our notation $S_{n,m}$ to the language of binomial coefficients with eq. (12):

$$S_{n,m+1} = \binom{n+m}{n}, \quad S_{n,m} + S_{n-1,m+1} = \binom{n+m-1}{n} + \binom{n+m-1}{n-1}$$

These are equal because of eq. (2). Or, you could just look at Figure 4 and justify it because of the way Pascal’s triangle was built. □

4 Even more triangles: Generalizing Pascal’s triangle to higher (and lower) dimensions

4.1 The multinomial theorem

When talking about binomial coefficients, we say $\binom{n}{k}$ is the number of ways to pick k things out of n things. A similar way to see this is the number of ways to organize n

things into two groups: One of size k and another one of size $n-k$. We can see that this is analogous by thinking of the group of $n-k$ things as the group of things "we don't pick" in the first definition.

This idea can be generalized into any number of groups. These are called *multinomial coefficients* and are defined as

$$\left(\binom{n}{k_1, k_2, \ldots, k_m}\right) := \frac{n!}{k_1! k_2! \cdots k_m!} \tag{16}$$

This denotes the number of ways to organize n things into m groups, where the ith group has k_i things. When we say "organize", we can't leave out any "thing" outside of a group, every "thing" has to go into a group. This is to say,

$$k_1 + k_2 + \cdots + k_m = n \tag{17}$$

This restriction makes it so that "we have $m-1$ degrees of freedom". If we know the first $m-1$ k_i terms, we know the last one:

$$k_m = n - k_1 - k_2 - \cdots - k_{m-1}$$

Notice then, that the case $m = 2$ corresponds to our familiar binomial coefficients:

$$\left(\binom{n}{k_1, k_2}\right) = \frac{n!}{k_1! k_2!} = \frac{n!}{k_1!(n-k_1)!} = \binom{n}{k_1}$$

The symmetry of binomial coefficients is generalized in the multinomial case into the property that the order of the k_i terms doesn't matter, which is obvious because of multiplication's commutativity.

With this tool, we can state the *multinomial theorem*: A way to generalize the binomial theorem and write an expression for "Newton's polynomial". The multinomial theorem states that

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{k_1 + \cdots + k_m = n} \left(\binom{n}{k_1, \ldots, k_m}\right) a_1^{k_1} \cdots a_m^{k_m} \tag{18}$$

The underscript of the sum symbol here might be unfamiliar. Instead of iterating a term from an initial value to a final value, we are iterating through all the possible values k_i such that their sum is n.

Let's check this for $m = 3$ and $n = 2$. We can compute in a minute that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

The possible values for (k_1, k_2, k_3) in the sum in eq. (18) are:

- One of the values is 2 and the rest are 0. These are $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$. Their multinomial coefficient is $\left(\binom{2}{2,0,0}\right) = \frac{2!}{2!} = 1$.
- Two of the values are 1 and the third one is 0. These are $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. Their multinomial coefficient is $\left(\binom{2}{1,1,0}\right) = \frac{2!}{1!1!} = 2$.

This matches with our calculation.

We can also generalize eq. (5) in a similar way:

$$\sum_{k_1+\cdots+k_m=n} \left(\!\!\binom{n}{k_1,\ldots,k_m}\!\!\right) = m^n \tag{19}$$

Which is also true because $(1+1+\cdots+1)^n$ with m ones is m^n.

4.2 Pascal's tetrahedron

How many terms are in the sum in eq. (18)? This is an interesting combinatorics question. It would be the same number of possible values for $k_1, \ldots, k_m$ given the restriction (17). This is the number of ways to write n as a sum of m terms, which is known as a *weak m-composition* of n. We say it is *weak* because we allow the terms to be 0.

The not-weak m-compositions of n are $\binom{n-1}{m-1}$. You can see this is true by imagining that we are distributing $k-1$ addition symbols in-between n ones, so there are $n-1$ spaces to put them in. We can then deduce that the weak m-compositions of n are the same as the not-weak m-composition of $n+m$: We just remove 1 from each of the m summands that add up to $n+m$ and we get a weak m-composition of n. So, the weak m-composition of n are $\binom{n+m-1}{m-1}$. This is the number of terms in eq. (18).

Notice that this number is $S_{m-1,n+1}$, an $(m-1)$-simplex number. This hints at a possible way to organize the terms in the multinomial expansion. Instead of traditionally writing them in one line, let's write them as the dots in an $(m-1)$-simplex. The $m=1$ case is trivial: $a^n = 1a^n$, which is like writing the terms in a single point, the 0-simplex. For $m=2$, we write them in a line, the old way, like we did back in section 3.1. For $m=3$, we can write them in a triangle, with this new fancy notation:

$$(a+b+c)^0 = \sum \left\{ 1a^0b^0c^0 \right\}$$

$$(a+b+c)^1 = \sum \left\{ \begin{array}{ll} 1a^1b^0c^0 & \\ 1a^0b^0c^1 & 1a^0b^1c^0 \end{array} \right\}$$

$$(a+b+c)^2 = \sum \left\{ \begin{array}{lll} 1a^2b^0c^0 & & \\ 2a^1b^0c^1 & 2a^1b^1c^0 & \\ 1a^0b^0c^2 & 2a^0b^1c^1 & 1a^0b^2c^0 \end{array} \right\}$$

$$(a+b+c)^3 = \sum \left\{ \begin{array}{llll} 1a^3b^0c^0 & & & \\ 3a^2b^0c^1 & 3a^2b^1c^0 & & \\ 3a^1b^0c^2 & 6a^1b^1c^1 & 3a^1b^2c^0 & \\ 1a^0b^0c^3 & 3a^0b^1c^2 & 3a^0b^2c^1 & 1a^0b^3c^0 \end{array} \right\}$$

$$(a+b+c)^4 = \sum \left\{ \begin{array}{lllll} 1a^4b^0c^0 & & & & \\ 4a^3b^0c^1 & 4a^3b^1c^0 & & & \\ 6a^2b^0c^2 & 12a^2b^1c^1 & 6a^2b^2c^0 & & \\ 4a^1b^0c^3 & 12a^1b^1c^2 & 12a^1b^2c^1 & 4a^1b^3c^0 & \\ 1a^0b^0c^4 & 4a^0b^1c^3 & 6a^0b^2c^2 & 4a^0b^3c^1 & 1a^0b^4c^0 \end{array} \right\}$$

Notice how the powers of a a, b or c rise and fall as you walk towards and away the corner with a^n, b^n or c^n respectively.

These triangles seem to behave awfully like the rows in Pascal's triangle. The thing is, we can build something like Pascal's triangle, but in 3 dimensions: *Pascal's tetrahedron* or *Pascal's pyramid*. Let's stack these coefficient triangles into a pyramid. Drawing this on paper would be pretty unreadable, so let's write it "floor by floor":

$n=0$	1				
$n=1$	1				
	1	1			
$n=2$	1				
	2	2			
	1	2	1		
$n=3$	1				
	3	3			
	3	6	3		
	1	3	3	1	
$n=4$	1				
	4	4			
	6	12	6		
	4	12	12	4	
	1	4	6	4	1

$$\vdots$$

Figure 5: First few "floors" of a sliced Pascal's tetrahedron

Like the triangle, Pascal's tetrahedron can also be built recursively by hand. It is a bit hard to see in Figure 5 because of the way the numbers are justified to the left. In the same way that a number in Pascal's triangle is the sum of the two above it, a number in Pascal's tetrahedron is the sum of the three above it. Let's color some of these to see this phenomenon:

$n=0$	1				
$n=1$	1				
	1	1			
$n=2$	1				
	2	2			
	1	2	1		
$n=3$	1				
	3	3			
	3	6	3		
	1	3	3	1	
$n=4$	1				
	4	4			
	6	12	6		
	4	12	12	4	
	1	4	6	4	1

$\vdots$

Figure 6: Sliced Pascal's tetrahedron with colors to indicate the sums.

The blue numbers in "floor" $n = 3$ add up to twelve, which gets passed down to $n = 4$. The same happens in $n = 1$, but now one of the three red numbers is not there. It is "outside the pyramid", so we can think of it as a 0, like we did with Pascal's triangle. The red numbers add up to 2, so it gets passed down to $n = 2$.

What is going on can be written as

$$\left(\!\!\binom{n}{k_1, k_2, k_3}\!\!\right) = \left(\!\!\binom{n-1}{k_1-1, k_2, k_3}\!\!\right) + \left(\!\!\binom{n-1}{k_1, k_2-1, k_3}\!\!\right) + \left(\!\!\binom{n-1}{k_1, k_2, k_3-1}\!\!\right) \tag{20}$$

Again, we prove this like we proved eq. (2):

$$\left(\!\!\binom{n-1}{k_1-1, k_2, k_3}\!\!\right) + \left(\!\!\binom{n-1}{k_1, k_2-1, k_3}\!\!\right) + \left(\!\!\binom{n-1}{k_1, k_2, k_3-1}\!\!\right) =$$
$$= \frac{(n-1)!}{(k_1-1)!k_2!k_3!} + \frac{(n-1)!}{k_1!(k_2-1)!k_3!} + \frac{(n-1)!}{k_1!k_2!(k_3-1)!} =$$
$$= \frac{(n-1)!k_1}{k_1!k_2!k_3!} + \frac{(n-1)!k_2}{k_1!k_2!k_3!} + \frac{(n-1)!k_3}{k_1!k_2!k_3!} = \frac{(n-1)!(k_1+k_2+k_3)}{k_1!k_2!k_3!} =$$
$$= \frac{n!}{k_1!k_2!k_3!} = \left(\!\!\binom{n}{k_1, k_2, k_3}\!\!\right) \quad \square$$

Let's take a look at another interesting fact: The edges of each of these triangular floors are the rows of Pascal's triangle. This is immediate mathematically, since

$$\left(\!\!\binom{n}{k_1, k_2, 0}\!\!\right) = \frac{n!}{k_1!k_2!0!} = \frac{n!}{k_1!k_2!} = \binom{n}{k_1}$$

More interesting is that the rows of this floor (and columns and diagonals) are multiples of rows of Pascal's triangle. In particular, they're multiplied by the number at the edges. This is because

$$\left(\!\!\binom{n}{k_1, k_2, k_3}\!\!\right) = \frac{n!}{k_1!k_2!k_3!} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!k_3!} = \binom{n}{k_1}\binom{n-k_1}{k_2} \qquad (21)$$

This is a very useful way to build Pascal's tetrahedron without doing the additions that need some 3D spatial imagination. Figure 7 shows this process.

4.3 Pascal's m-simplex

If we can have Pascal's triangle and Pascal's tetrahedron, are there more Pascal's "shapes"? Before going up another dimension, let's quickly look at what a 1D Pascal's triangle looks like. If we go by the same logic of starting with a 1 and adding the m numbers above to get the next number, a 1-dimensional Pascal's triangle, *Pascal's line* should be just a line of ones. This follows the multinomial theorem:

$$a^n = \sum_{k_1=n} \left(\!\!\binom{n}{k_1}\!\!\right) a^n = \frac{n!}{n!}a^n = a^n$$

Now, the 0D Pascal's triangle, *Pascal's point*. Again, by the same construction process, it should start with a 1. And that should be it. A single 1. The multinomial theorem also holds, since the *empty sum* is 0, the additive identity, by convention and both sides in eq. (18) are empty sums. This sounds right, but the only floor on Pascal's point is $n = 0$, and in this case, the left side of eq. (18) is 0^0. This power is sometimes undefined. However, by convention (again) it is usually defined as $0^0 = 1$ in the context of combinatorics. So, the multinomial theorem doesn't hold? Well, maybe the right side of eq. (18) isn't an empty sum. The same way that we count the 0-combination as one combination, there is one way for no k_i terms to add up to $n = 0$. So, what is the multinomial with no k_i terms? It's denominator is an empty product, so we get that

$$\left(\!\!\binom{0}{\cdot}\!\!\right) = \frac{0!}{1} = 0! = 1$$

$n=0$	1	1	1
$n=1$	1	1	1
	1	1 1	1 1
$n=2$	1	1	1
	2	1 1	2 2
	1	1 2 1	1 2 1
$n=3$	1	1	1
	3	1 1	3 3
	3	1 2 1	3 6 3
	1	1 3 3 1	1 3 3 1
$n=4$	1	1	1
	4	1 1	4 4
	6	1 2 1	6 12 6
	4	1 3 3 1	4 12 12 4
	1	1 4 6 4 1	1 4 6 4 1

$$\vdots$$

Figure 7: The floors of Pascal's tetrahedron can be obtained by "multiplying Pascal's triangle by its rows".

$$0^0 = 1 = \sum_{0=0} \left(\binom{0}{.}\right) = 1$$

This is very weird notation, but it makes a bit of sense.

$$1 \quad 1 \quad 1 \quad 1 \quad \cdots \qquad\qquad\qquad\qquad 1$$

Figure 8: Pascal's line and point

Ok so, higher dimensions, right? *Pascal's m-simplex.* We start with a one and every number after it is the sum of the m numbers above it. Every *floor* is an $(m-1)$-simplex, filled with multinomial coefficients of the form

$$\mathcal{P}^n_{k_1,\ldots,k_m} := \left(\binom{n}{k_1,\ldots,k_m}\right) \tag{22}$$

We won't be taking powers of these terms, so we won't be confused with the superscript notation. For the $m = 2$ case, we have $\mathcal{P}^n_{k_1,k_2} = P_{n,k_1}$. We see again that this definition is compatible with the recurrence rule:

$$\mathcal{P}^n_{k_1,k_2,\dots,k_m} = \mathcal{P}^{n-1}_{k_1-1,k_2\dots,k_m} + \mathcal{P}^{n-1}_{k_1,k_2-1,\dots,k_m} + \cdots + \mathcal{P}^{n-1}_{k_1,k_2,\dots,k_m-1} \tag{23}$$

And we prove it again in a similar fashion:

$$\begin{aligned}
&\mathcal{P}^{n-1}_{k_1-1,k_2\dots,k_m} + \mathcal{P}^{n-1}_{k_1,k_2-1,\dots,k_m} + \cdots + \mathcal{P}^{n-1}_{k_1,k_2,\dots,k_m-1} = \\
&= \frac{(n-1)!}{(k_1-1)!\cdots k_m!} + \cdots + \frac{(n-1)!}{k_1!\cdots(k_m-1)!} = \sum_{i=1}^{m} \frac{(n-1)!k_i}{k_1!\cdots k_m!} = \\
&= \frac{(n-1)!(k_1+\cdots k_m)}{k_1!\cdots k_m!} = \frac{(n-1)!n}{k_1!\cdots k_n!} = \frac{n!}{k_1!\cdots k_n!} = \left(\binom{n}{k_1,\dots,k_m}\right) = \mathcal{P}^n_{k_1,\dots,k_m} \quad \square
\end{aligned}$$

Eq. (21) can also be generalized:

$$\mathcal{P}^n_{k_1,k_2,\dots,k_m} = \binom{n}{k_1}\mathcal{P}^{n-k_1}_{k_2,\dots,k_m} \tag{24}$$

Applying this property again and again, we get that

$$\begin{aligned}
\mathcal{P}^n_{k_1,k_2,\dots,k_m} &= \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{m-1}}{k_{m-1}}\mathcal{P}^{n-k_1-\cdots-k_{m-1}}_{k_m} = \\
&= \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{m-1}}{k_{m-1}}\mathcal{P}^{k_m}_{k_m} = \\
&= \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{m-1}}{k_{m-1}}
\end{aligned} \tag{25}$$

This property is reflected in Figure 9, where each floor is a tetrahedron, but you can see that each floor's "slice" is a multiple of a floor of Pascal's tetrahedron (Figure 5). In particular, it is multiplied by $\binom{n}{s}$. Recall too how the rows of the floors of the tetrahedron were multiples of Pascal's triangle's rows.

	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$n=0$	1				
$n=1$	1 1 1	1			
$n=2$	1 2 2 1 2 1	2 2 2	1		
$n=3$	1 3 3 3 6 3 1 3 3 1	3 6 6 3 6 3	3 3 3	1	
$n=4$	1 4 4 6 12 6 4 12 12 4 1 4 6 4 1	4 12 12 12 24 12 4 12 12 4	6 12 12 6 12 12	4 4 4	1

$\vdots$

Figure 9: First few floors of Pascal's pentachoron. Each floor is a tetrahedron, so it's been sliced with an slice per column. Four numbers have been colored in floor $n=3$ and their sum is passed down to floor $n=4$.